

AS-ITP-94-62

November, 1994

# Cyclic Quantum Dilogarithm and Shift Operator <sup>1</sup>

**Zhan-Ning Hu** <sup>2</sup>

CCAST(WORLD LABORATORY) P.O.BOX 8730, BEIJING, 100080

and

INSTITUTE OF THEORETICAL PHYSICS, ACADEMIA SINICA,

P. O. BOX 2735 BEIJING 100080, CHINA <sup>3</sup>

**Bo-Yu Hou**

INSTITUTE OF MODERN PHYSICS, NORTHWEST UNIVERSITY, XIAN, CHINA

## Abstract

From the cyclic quantum dilogarithm the shift operator is constructed with  $q$  is a root of unit and the representation is given for the current algebra introduced by Faddeev *et al.* It is shown that the theta-function is factorizable also in this case by using the star-square equation of the Baxter-Bazhanov model.

PACS. 11.10 - Field theory.

PACS. 02.10 - Algebraic methods.

PACS. 05.50 - Lattice theory and statistics.

---

<sup>1</sup>This research was partially supported by the China center of advanced science and technology.

<sup>2</sup>email address: huzn@itp.ac.cn

<sup>3</sup>mail address

# 1 Introduction

Recently people have pay more attention to the investigation of completely integrable discrete variants of conformal field theory such as lattice Wess-Zumino-Witten model [1], quantum Volterra model [2], the lattice Liouville model and the lattice sine-Gordon model [3]. Bruschi *et al* studied the properties and behaviors of integrable symplectic maps from the lattice evolution equation[4]. In order to providing the new way to construct the conservation laws and to discuss the nature of quantum integrability by using the discrete space-time method [5, 6] Faddeev *et al* investigated the lattice Virasoro algebra and the corresponding hierarchy of conservation laws from the lattice current algebra where the shift operator with  $|q| < 1$  is discussed in detail [7].

We know that the dilogarithm appeared firstly in the discussion of the  $XXZ$ -model at small magnetic field [8], then appeared in the 2-dimensional quantum field theories and solvable lattice models [9]. The quantum dilogarithm identity introduced in Ref. [10] is equivalent to the restricted star-triangle relation of the three-dimensional Baxter-Bazhanov model [11, 12] and more recently Kashaev built a connection between cyclic  $6j$ -symbol and the quantum dilogarithm.

In this letter, we will construct the shift operator from the cyclic quantum dilogarithm and give a representation of the lattice current algebra which combined with free discrete time dynamics. It is proved that the “theta-function” is also factorizable in the case of  $|q| = 1$  by using the star-square equation [13, 14] of the three-dimensional Baxter-Bazhanov model. In section 2 we give the cyclic quantum dilogarithm and the representation of the lattice current algebra. The shift operator is constructed and the “theta-function” is proved to be factorizable in section 3. Finally some remarks are given.

## 2 Cyclic Quantum Dilogarithm and the Lattice Current Algebra

By following the Refs.[11, 13], define the function

$$w(a, b, c|l) = \prod_{j=1}^l b/(c - a\omega^j), \quad a^L + b^L = c^L, \quad l \geq 0, \quad (1)$$

with  $w(a, b, c|0) = 1$ . For any operator  $A$  whose  $L$ -th power is the identity operator, the spectrum of the operator  $A$  is given by  $L$  distinct numbers

$$\omega^l, \quad l = 0, 1, \dots, L-1. \quad (2)$$

We define the cyclic quantum dilogarithms  $\Psi(A)$  and  $\Phi(A)$  (See Ref. [10]) are the commuted operators which depend on the operator  $A$  and commute with  $A$  which has the spectrum (2). The spectrums of  $\Psi(A)$  and  $\Phi(A)$  have the following forms:

$$\begin{aligned} \Psi(\omega^l) &= \Psi(1)w(a, b, c|l), \\ \Phi(\omega^l) &= \Phi(1)w(c, \omega^{1/2}b, \omega a|l), \end{aligned} \quad (3)$$

where  $\Psi(1)$  and  $\Phi(1)$  are the non-zero complex factors. And the “functional” relations of the cyclic quantum dilogarithms  $\Psi(A)$  and  $\Phi(A)$  can be written as

$$\begin{aligned} \Psi(\omega^{-1}A)\Psi(A)^{-1} &= (c - aA)/b, \\ \Phi(\omega^{-1}A)\Phi(A)^{-1} &= (\omega^{1/2}a - \omega^{-1/2}cA)/b, \end{aligned} \quad (4)$$

which determine the operators  $\Psi(A)$  and  $\Phi(A)$  up to the complex factors. Furthermore, from the above relations, we have

$$\Psi(A) = \rho_1 \sum_{l=0}^{L-1} A^l \prod_{j=1}^l \frac{a}{c - b\omega^{-j}}, \quad (5)$$

$$\Phi(A) = \rho_2 \sum_{l=0}^{L-1} A^l \prod_{j=1}^l \frac{c}{a\omega - b\omega^{1/2-j}}, \quad (6)$$

where  $\rho_1$  and  $\rho_2$  are also the non-zero complex factors. In this way,  $\Psi(1)$  and  $\Phi(1)$  can be expressed as

$$\begin{aligned}\Psi(1) &= \rho_1 \frac{c - a\omega}{b} \sum_{l=0}^{L-1} \prod_{j=1}^l \frac{a\omega}{c - b\omega^{-j}}, \\ \Phi(1) &= \rho_2 \frac{\omega^{1/2}(a - c)}{b} \sum_{l=0}^{L-1} \prod_{j=1}^l \frac{c\omega}{a\omega - b\omega^{1/2-j}}.\end{aligned}\tag{7}$$

By using the cyclic quantum dilogarithms (4) we will construct the shift operator with  $|q| = 1$ , in the following section, from the lattice current algebra

$$\begin{aligned}w_{n-1}w_n &= q^2 w_n w_{n-1}, \quad n = 2, 3, \dots, 2N, \\ w_{2N}w_1 &= q^2 w_1 w_{2N}, \\ w_m w_n &= w_n w_m, \quad 1 < |m - n| < 2N - 1,\end{aligned}\tag{8}$$

which reduces to the periodic free field in the continuous limit [7]. Set

$$\begin{aligned}x_i &= \underbrace{I \otimes I \otimes \dots \otimes I}_{i-1} \otimes x \otimes \underbrace{I \otimes I \otimes \dots \otimes I}_{N-i-1}, \\ y_i &= \underbrace{I \otimes I \otimes \dots \otimes I}_{i-1} \otimes y \otimes \underbrace{I \otimes I \otimes \dots \otimes I}_{N-i-1},\end{aligned}\tag{9}$$

with  $i = 1, 2, \dots, N - 1$ , where the  $L$ -by- $L$  matrices  $x, y$  are given by

$$x = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{L-1} \end{bmatrix},\tag{10}$$

with  $\omega = \exp(2\pi i/L)$  and  $I$  is the  $L$ -by- $L$  unit matrix. When we fix that  $w_n^L = 1$  ( $n = 1, 2, \dots, 2N$ ) the representation of the lattice current algebra (8) with  $q^2 = \omega$  is given from the following relations:

$$\begin{aligned}w_{2i+1} &= x_i^{-1} x_{i+1}, \quad i = 0, 1, 2, \dots, N - 1, \\ w_{2j} &= y_j, \quad j = 1, 2, \dots, N - 1, \\ w_{2N} &= \prod_{l=1}^{N-1} y_l^{-1},\end{aligned}\tag{11}$$

where  $x_0 = x_N = 1$  and  $x_i, y_i$  ( $i = 1, 2, \dots, N - 1$ ) are the  $N - 1$  independent Weyl pairs satisfied the relations  $x_i y_i = q^2 y_i x_i$ .

### 3 Shift Operator and Factorized “Theta Function”

In the discrete space-time picture the equation of motion of the free field  $p(x)$ ,  $p_t(x, t) + p_x(x, t) = 0$ , has the form  $w_n(t+1) = w_{n-1}(t)$  with  $w_n(0) = w_n$ . Then it exists that the shift operator which satisfy

$$w_n U = U w_{n-1}. \quad (12)$$

The main aim in the rest of this letter is to discuss it by using the cyclic quantum dilogarithms. Let

$$h_n = \Psi(w_n) \Phi(w_n^{-1}) \quad (13)$$

with  $w_n^L = 1$  for  $n = 1, 2, \dots, 2N$ . From relation (4) we have

$$w_n h_{n-1} = -\omega^{1/2} h_{n-1} w_n w_{n-1}, \quad h_n w_{n-1} = -\omega^{1/2} w_n w_{n-1} h_n. \quad (14)$$

It can be obtained easily that

$$w_n h_{n-1} h_n = h_{n-1} h_n w_{n-1}, \quad n = 1, 2, 3, \dots, 2N, \quad (15)$$

with  $w_0 = w_{2N}$ . By considering that the two central elements  $C_1 = \prod w_{odd}$  and  $C_2 = \prod w_{even}$  are equal <sup>4</sup>, from the above relation, we can express the shift operator  $U$  as

$$U = h_1 h_2 \cdots h_{2N-1}, \quad (16)$$

which satisfies the relation (12). And operators  $h_n$  give a representation of the braid group of the  $A_{2N-1}^{(1)}$  type. It is interesting that the above relations hold also when we substitute operators  $h_n$  by  $\bar{h}_n = \Phi(w_n) \Psi(w_n^{-1})$ . In the other hand, the shift operator with  $|q| \neq 1$  can be denoted by the theta functions [10, 15]. Then we can ask the questions that what is the difference between the operators  $h_n$  and  $\bar{h}_n$  and what happened about the theta functions when it denotes the shift operator with

---

<sup>4</sup>We can always get the lattice current algebra with the two equal central elements  $C_1$  and  $C_2$  by choosing  $w_n$  properly.

$|q| = 1$ . The answers of them are given as the follows. We know that the star-square relation of the three dimensional Baxter-Bazhanov model can be written as [13, 14]

$$\left\{ \sum_{\sigma \in Z_N} \frac{w(x_1, y_1, z_1|a + \sigma)w(x_2, y_2, z_2|b + \sigma)}{w(x_3, y_3, z_3|c + \sigma)w(x_4, y_4, z_4|d + \sigma)} \right\}_0$$

$$= \frac{(x_2 y_1 / x_1 z_2)^a (x_1 y_2 / x_2 z_1)^b (z_3 / y_3)^c (z_4 / y_4)^d}{\gamma(a - b) \omega^{(a+b)/2}}$$

$$\times \frac{w(\omega x_3 x_4 z_1 z_2 / x_1 x_2 z_3 z_4 | c + d - a - b)}{w\left(\frac{x_4 z_1}{x_1 z_4} | d - a\right) w\left(\frac{x_3 z_2}{x_2 z_3} | c - b\right) w\left(\frac{x_3 z_1}{x_1 z_3} | c - a\right) w\left(\frac{x_4 z_2}{x_2 z_4} | d - b\right)}, \quad (17)$$

with the constraint condition  $y_1 y_2 z_3 z_4 / (z_1 z_2 y_3 y_4) = \omega$  where the subscript "0" after the curly brackets indicates that the l. h. s. of the above equation is normalized to unity at zero exterior spins and the following notations are used:

$$w(x, y, z|l) = (y/z)^l w(x/z|l), \quad \gamma(a - b) = \omega^{(a-b)(L+a-b)/2}. \quad (18)$$

From Eqs. (5), (6) and (7), the operators  $h_n$  can be denoted as

$$h_n = \rho_1 \rho_2 \sum_{k=0}^{L-1} a_k w_n^k \quad (19)$$

where

$$a_k = \sum_{l=0}^{L-1} \frac{1}{w(\omega^{-1}b, a, c|l)w(\omega^{-1/2}b, c, \omega a|k + l)}. \quad (20)$$

By using the above star-square relation it can be proved that

$$a_k = \gamma(k) a_0 \quad (21)$$

where  $\gamma(k)$  is given in Eq. (18). Then we have

$$h_n = a_0 \rho_1 \rho_2 \theta(w_n), \quad \theta(w_n) = \sum_{k=0}^{L-1} (-)^k \omega^{k^2/2} w_n^k. \quad (22)$$

Furthermore we can proved that  $h_n = \bar{h}_n$  by using Eqs. (17) and (21). Therefore the shift operator is constructed from the cyclic quantum dilogarithms by which the "theta function" is factorized when  $|q| = 1$ .

## 4 Conclusions and Remarks

periodic free field in the discrete time-space picture is constructed from the lattice current algebra for which the cyclic representation is given from  $N - 1$  independent Weyl pairs. And we show that the “theta function” is also factorizable when  $|q| = 1$  by using the star-square relation of the three dimensional Baxter-Bazhanov model.

In fact, the “shift” property can be connected to many domains of physics such as the quantum field theory, the solvable models of statistical mechanics and the dynamical systems. Recently Faddeev and Volkov discussed the shift of the saw  $S$  corresponding to the solution of Hirota Equation along the discrete time axis as an example of an integrable symplectic map [16]. The shift operator also appeared in the differential forms when we set  $\partial_r f = f - R_r f$ ,  $R_r f(k) = f(k + 1)$ , then  $f(k + 1)X = Xf(k)$  on the lattice where  $X$  is 1-form and it will be useful in the lattice gauge theory [17]. Another line of the developments is in the domain of the supermanifolds and it is not very clear about the structure of it when the relations similar to the lattice current algebra is introduced <sup>5</sup>. Due to the cyclic quantum dilogarithms have the deep connections with the solved lattice models we can discuss the shift properties of the two and three dimensional lattice model further.

## Acknowledgment

One of the authors (Hu) would like to thank H. Y. Guo, K. Wu for the helpful discussions and B. Yua. Hou, Z. Q. Ma, K. J. Shi, X. C. Song , S. Y. Zhou and P. Wang for the interests in this job at the workshop of CCAST(World Laboratory) in Beijing.

## References

---

<sup>5</sup>One of the authors (Hu) would like to thank Prof. D. A. Leites, University of Stockholm, for the report about supermanifolds in Nov, 1994.

- [1] Falceto F. and Gawedzki K., *J. Geo. Phys.*, **11** (1993) 251.
- [2] Volkov A. Yu., *Phys. Lett. A*, **167** (1992) 345.
- [3] Faddeev L. D. and Takhtajan L. A., *Lect. Notes in Phys.*, **246** (1986) 166.  
       Gervais J. -L., *Phys. Lett. B*, **160** (1985) 277; **160** (1985) 279.  
       Volkov A. Yu., *Theor. Math. Phys.*, **74** (1988) 135.  
       Babelon O., *Phys. Lett. B*, **238** (1990) 234.
- [4] Bruschi M., Ragnisco O., Santini P. M. and Tu G. Z., Integrable Symplectic Maps, preprint, (1990).
- [5] Destri C. and de Vega H. J., *Nucl. Phys. B* **290** (1987) 363; *J. Phys. A*, **22** (1989) 1329.
- [6] Nijhoff F. W., Capel H. W. and Papageorgiu V. G., *Phys. Rev. A*, **46** (1992) 2155; Nijhoff F. W. and Capel H. W., *Phys. Lett. A*, **163** (1992) 49.
- [7] Faddeev L. and Volkov A. Yu., Abelian current algebra and the Virasoro algebra on the lattice, preprint, HU-TFT-93-29\* (1993).
- [8] Bazhanov V. and N. Reshetikhin, *J. Phys. A*, **23** (1990) 1477.
- [9] Klümper A, Batchelor M. T. and Pearce P. A., *J. Phys. A* **24** (1991) 3111.  
       Martins M. J., *Phys. Rev. Lett.* **65** (1990) 2091.  
       Fateev V. A. and Zamolodchikov A. B., *Phys. Lett. B*, **271** (1991) 91.  
       Klassen T. R. and Melzer E., *Nucl. Phys. B*, **338** (1990) 485; **350** (1991) 635.
- [10] Faddeev L. D. and Kashaev R. M., Quantum Dilogarithm, Preprint, September (1993).
- [11] Bazhanov V. V. and Baxter R. J., *J. Stat. Phys.*, **69** (1992) 453; **71** (1993) 839.
- [12] Hu Z. N., Three-Dimensional Star-Star Relation, *Int. J. Mod. Phys. A* (to appear).



- [13] Kashaev R. M., Mangazeev V. V. and Stroganov Yu. G., *Int. J. Mod. Phys. A*, **8** (1993) 1399.
- [14] Hu Z. N. , *Mod. Phys. Lett. B*, **8** (1994) 779.
- [15] Hu Z. N., Weyl Pair, Current Algebra and Shift Operator, *Phys. Lett. A* (to appear).
- [16] Faddeev L. and Volkov A. Yu., *Lett. Math. Phys.*, **32** (1994) 125.
- [17] Wilson K. G., *Phys. Rev. D*, **10** (1974) 2445.